ON A THEOREM OF CONANT-VOGTMANN

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ABSTRACT. We prove that the graph complex is a strong homotopy Lie (super) bialgebra.

1. Introduction and definitions

The graph complex was introduced by Kontsevich in [K1] and [K2]. In [CV1], Conant and Vogtmann constructed a new differential, a Lie bracket and a Lie cobracket on the graph complex. They proved that on the subspace of one-particle irreducible graphs, the Lie bracket and the Lie cobracket give a Lie bialgebra structure. In this paper, we prove that the whole graph complex is a *strong homotopy Lie (super) bialgebra* (cf. [M, §5], [G, §3.2]).

We shall work in the category of super vector spaces $V = V_+ \oplus V_-$ over a field of characteristic 0. Denote by Π the parity-change functor $(\Pi V)_\pm = V_\mp$.

By a graph, we mean a finite 1-dimensional CW-complex (not necessarily connected) such that the valency of each vertex is at least 3. An orientation or of a graph X is an ordering of its vertices and a choice of direction on each of its edges (cf. [CV2, Definition 2]). Switching the order of two vertices or reversing the direction of an edge changes the orientation from or to $-\mathsf{or}$. We shall identify an ordering of the vertices with a labeling of the vertices by $1, 2, \ldots$.

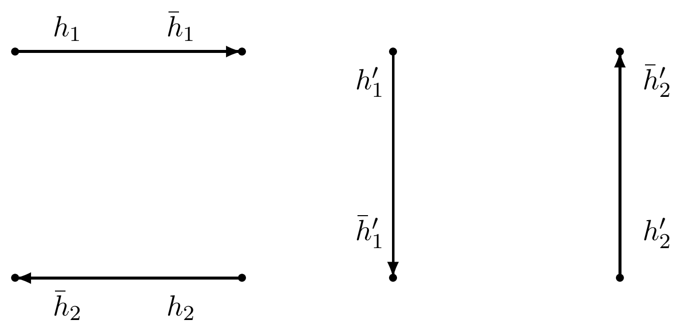
Let $\mathfrak{G}$ be the vector space generated by graphs (X, or), modulo the equivalence relation $(X, -\mathsf{or}) = -(X, \mathsf{or})$. Observe that if X has an edge-loop, then it is equal to 0 in $\mathfrak{G}$. Let $\mathfrak{G}_+$ (resp. $\mathfrak{G}_-$) be the subspace of $\mathfrak{G}$ spanned by graphs with an even (resp. odd) number of vertices. Clearly, $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$. If $X, Y \in \mathfrak{G}$, then define their product $X \cdot Y$ to be the disjoint union of X and Y. The orientation of $X \cdot Y$ is obtained by adding the number of vertices of X to the labels of the vertices of Y. The vector space $\mathfrak{G}$ is a commutative super algebra by extending this product bilinearly to linear combinations of graphs.

Given an edge e of a graph $X \in \mathfrak{G}$ such that e is not an edge-loop, we denote by X_e the graph obtained from X by contracting the edge e. The induced orientation on X_e is defined as follows: if the source vertex of e is labeled 1 and the target vertex of e is labeled 2, then the vertex of X_e formed by the contracted edge e is labelled 1, and the labeling of other vertices are reduced by 1. If e is an edge-loop, then we set X_e to be 0.

Given a half-edge h of a graph $X \in \mathfrak{G}$, we denote by $v(h)$ the vertex of X attached to h, and by $\bar{h}$ the half-edge such that $e(h) := h \cup \bar{h}$ is an edge of X.

Let h_1, h_2 be two half-edges such that they belong to two distinct edges of X, and choose a representative for the orientation on X so that $v(h_1)$ (resp. $v(h_2)$) is the source vertex of $e(h_1)$ (resp. $e(h_2)$). Assume that $e(h_1)$ and $e(h_2)$ are not edge-loops. Denote by $X\langle h_1, h_2\rangle$ the graph obtained from X by adding in an edge e_1 directed from $v(h_1)$ to $v(\bar{h}_2)$, adding in an edge e_2 directed from $v(h_2)$ to $v(\bar{h}_1)$, and deleting away the two edges $e(h_1), e(h_2)$. Denote by h'_i $(i = 1, 2)$ the half-edge of $X\langle h_1, h_2\rangle$ such that $e_i = e(h'_i)$ and e_i is directed from h'_i to $\bar{h}'_i$; see following picture:

In X: In $X\langle h_1, h_2\rangle$:



Finally, denote $X_{h_1,h_2} := X\langle h_1, h_2\rangle_{e_1}$. If v is a vertex of $X\langle h_1, h_2\rangle$ attached to e_1, we will also denote by v the vertex of X_{h_1,h_2} contracted from the edge e_1. (Thus, the vertex of X_{h_1,h_2} contracted from e_1 may be denoted by more than one name.) Observe that $X_{\bar{h}_2,\bar{h}_1} = X_{h_1,h_2}$. If $e(h_1)$ or $e(h_2)$ is an edge-loop, then we set X_{h_1,h_2} to be 0. (Our notation differs from the one in [CV1], where they write $X_{h_1\bar{h_2}}$ for our X_{h_1,h_2}.)

For positive integers $m, n \leq 2$, we define an odd linear map $\alpha_{m,n} : \mathfrak{G}^{\otimes n} \to \mathfrak{G}^{\otimes m}$ as follows:

$$\alpha_{m,n}(X_1 \otimes \cdots \otimes X_n) := \sum_{h_1,h_2} \sum_{Y_1\otimes\cdots\otimes Y_m} Y_1 \otimes \cdots \otimes Y_m. \tag{1}$$

In (1), the first summation is taken over all ordered pairs of half-edges h_1, h_2 of $X_1 \cdots X_n$ such that each graph X_i contains h_j for some j, and h_1, h_2 belong to distinct edges. The second summation is taken over all $Y_1 \otimes \cdots \otimes Y_m \in \mathfrak{G}^{\otimes m}$ such that

$$Y_1 \cdots Y_m = (X_1 \cdots X_n)_{h_1,h_2}, \tag{2}$$

and each Y_i is a graph containing $v(h'_j)$ for some j. If m and n are positive integers such that $m > 2$ or $n > 2$, then we let $\alpha_{m,n} : \mathfrak{G}^{\otimes n} \to \mathfrak{G}^{\otimes m}$ be the zero map. The maps $\alpha_{1,1}$, $\alpha_{1,2}$, and $\alpha_{2,1}$ are, respectively, the differential, Lie bracket, and Lie cobracket constructed in [CV1], but we will not use these facts in the proof of our theorem below. The map $\alpha_{2,2}$ is new.

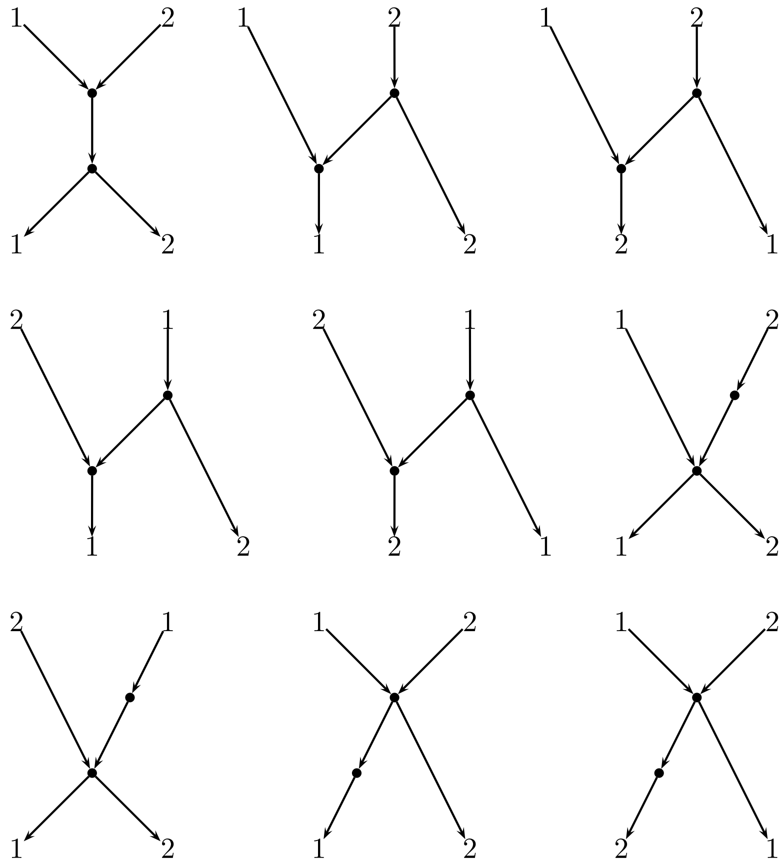


FIGURE 1. Elements of $T(2,2)$.

By a corolla, we mean a vertex with directed half-edges attached to it such that there is at least one incoming half-edge and at least one outgoing half-edge. Let m, n be positive integers. We shall denote by $T(m,n)$ the set consisting of all "flowcharts" T described as follows: T is obtained from two corollas, called $s(T)$ and $t(T)$, by joining an outgoing half-edge of $s(T)$ to an incoming half-edge of $t(T)$, such that the resulting flowchart has n inputs and m outputs, and moreover there is a labeling of the inputs by $1, \ldots, n$ and a labeling of the outputs by $1, \ldots, m$. For example, the elements of $T(2,2)$ are listed in Figure 1. If v is a corolla, we write $i(v)$ (resp. $o(v)$) for the number of incoming (resp. outgoing) half-edges at v. For each flowchart $T \in T(m,n)$, we write

$$\alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} : \mathfrak{G}^{\otimes n} \to \mathfrak{G}^{\otimes m}$$

for the composition of $\alpha_{o(t(T)),i(t(T))}$ and $\alpha_{o(s(T)),i(s(T))}$ according to T.

Theorem 3. *For each ordered pair of positive integers m, n, the map $\alpha_{m,n}$ is commutative and cocommutative, and the following identity holds:*

$$\sum_{T\in T(m,n)} \alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} = 0. \tag{4}$$

In other words, $\Pi\mathfrak{G}$ is a strong homotopy Lie (super) bialgebra.

We will prove Theorem 3 in section 2. In the appendix at the end of the paper, we give the definition of strong homotopy Lie bialgebra.

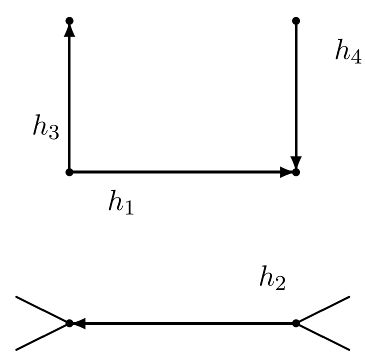
Note that the left hand side of (4) is zero if m or n is greater than 3 because $\alpha_{i,j} = 0$ if i or j is greater than 2. Taking $(m, n) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 1)$ in (4), we obtain the results of [CV1] that $\alpha_{1,1}$ is a differential, $\alpha_{1,1}$ is a derivation with respect to $\alpha_{1,2}$, $\alpha_{1,2}$ satisfies the Jacobi identity, $\alpha_{1,1}$ is a coderivation with respect to $\alpha_{2,1}$, and $\alpha_{2,1}$ satisfies the coJacobi identity. Taking $(m, n) = (2, 2)$, $(2, 3)$, $(3, 2)$ and $(3, 3)$, we obtain new identities. In particular, taking $(m, n) = (2, 2)$, we deduce [CV1, Theorem 1] which states that $\Pi\mathfrak{G}^{1PI}$ is a Lie super bialgebra, where $\mathfrak{G}^{1PI} \subset \mathfrak{G}$ is the subspace spanned by one-particle irreducible graphs (i.e. connected graphs which remain connected after the removal of any edge). Indeed, it is plain that $\alpha_{2,2}(X_1 \otimes X_2) = 0$ if X_1 or X_2 is in $\mathfrak{G}^{1PI}$.

We remark that in [C], Conant constructed a family of strong homotopy Lie algebra structures and strong homotopy Lie coalgebra structures on graph complexes associated to cyclic operads. We do not know if these structures also form strong homotopy Lie bialgebras.

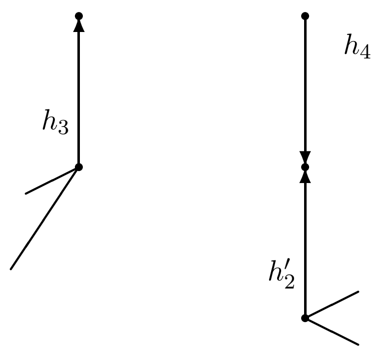
2. Proof of Theorem 3

The commutativity and cocommutativity of $\alpha_{m,n}$ are clear from (2). We have to prove that (4) holds. To this end, we shall pair the terms which appear in the left hand side of (4) such that the terms in a pair are negative of each other. Roughly speaking, the terms which come from two pairs of half-edges taken in different order have opposite signs. However, this is not always true, as we now illustrate:

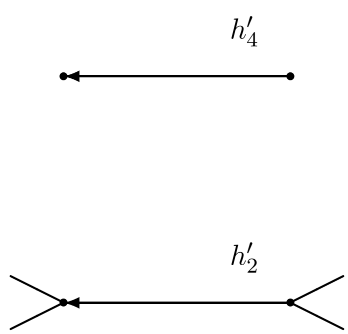
In X:



In X_{h_1,h_2}:



In $(X_{h_1,h_2})_{h_3,h_4}$:



However, in this example, the edge $e(h_1)$ becomes an edge-loop in X_{h_3,h_4} and hence X_{h_3,h_4} is equal to zero. In below, we shall pair the term $(X_{h_1,h_2})_{h_3,h_4}$ with $(X_{\bar{h}_1,h_2})_{h_3,h_4}$, which is equal to $-(X_{h_1,h_2})_{h_3,h_4}$ (the minus sign is due to orientation).

To make precise how the terms on the left hand side of (4) cancel, we shall define a set F and an involution $\mu : F \to F$. The terms in the left hand side of (4) will be grouped according to F, and the terms corresponding to $f \in F$ will cancel with the terms corresponding to $\mu(f) \in F$.

Fix any $m, n \leq 3$ and graphs $X_1, \ldots, X_n \in \mathfrak{G}$. We may assume that there is no edge-loop in $X_1 \cdots X_n$. Define F to be the set consisting of all data

$$f := (h_1, h_2, h_3, h_4, U_1, \ldots, U_m)$$

where

- h_1, h_2 are half-edges of $X_1 \cdots X_n$ which belong to distinct edges;
- $e(h_1'), e(h_2')$ are not edge-loops in $(X_1 \cdots X_n)\langle h_1, h_2\rangle$;

- h_3, h_4 are half-edges of $(X_1 \cdots X_n)_{h_1,h_2}$ which belong to distinct edges;
- $e(h_3), e(h_4)$ are not edge-loops in $(X_1 \cdots X_n)_{h_1,h_2}$, and $e(h'_3), e(h'_4)$ are not edge-loops in $(X_1 \cdots X_n)_{h_1,h_2}\langle h_3, h_4\rangle$;
- defining S_i $(i = 1, \ldots, n)$ to be the set of all $j \in \{1, 2, 3, 4\}$ such that either
  - X_i contains h_j, or
  - X_i contains h_2, and $h_j = h'_2$, or
  - X_i contains h_1, and $h_j = \overline{h'_2}$,

  we have: each S_i is nonempty and $S_1 \cup \ldots \cup S_n = \{1, 2, 3, 4\}$ is a disjoint union;
- at least one S_i which contains 1 or 2 also contains 3 or 4;
- each U_i is a nonempty subset of $\{1, 2, 3, 4\}$ and $U_1 \cup \ldots \cup U_m = \{1, 2, 3, 4\}$ is a disjoint union;
- at least one U_i which contains 3 or 4 also contains 1 or 2;
- if $v(h'_j)$, $v(h'_k)$ are in the same connected component of
$$((X_1 \cdots X_n)_{h_1,h_2})_{h_3,h_4}$$
  and U_i contains j, then U_i contains k.

Given $f \in F$ as above, there is a unique flowchart $T_f \in T(m, n)$ such that the input of T_f labelled i goes into $s(T_f)$ if and only if S_i contains 1 or 2, and the output of T_f labelled i goes out from $t(T_f)$ if and only if U_i contains 3 or 4.

Let $m, n \leq 3$. Consider the left hand side of (4) applied to $X_1 \otimes \cdots \otimes X_n$. For each $(h_1, h_2, h_3, h_4, U_1, \ldots, U_m) \in F$, there is a corresponding term

$$\sum_{Y_1 \otimes \cdots \otimes Y_m} Y_1 \otimes \cdots \otimes Y_m \tag{5}$$

where the summation is taken over all $Y_1 \otimes \cdots \otimes Y_m \in \mathfrak{G}^{\otimes m}$ such that $Y_1 \cdots Y_m = ((X_1 \cdots X_n)_{h_1,h_2})_{h_3,h_4}$ and Y_i is a graph containing $v(h'_j)$ if U_i contains j.

We define an involution $\mu : F \to F$ by

$$\mu(h_1, h_2, h_3, h_4, U_1, \ldots, U_m) := (h_1^\vee, h_2^\vee, h_3^\vee, h_4^\vee, U_1^\vee, \ldots, U_m^\vee),$$
$$U_i^\vee := \{j \mid \xi(j) \in U_i\}, \quad \xi : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\},$$

where $h_1^\vee, h_2^\vee, h_3^\vee, h_4^\vee$ and $U_1^\vee, \ldots, U_m^\vee$ will be defined accordingly in each of the following cases:

Case (i) , $h_3, h_4 \not\subseteq e(h_2)$:

- if $e(h_1)$ connects $v(h_3)$ with $v(\bar{h}_4)$, then
$$(h_1^\vee, h_2^\vee, h_3^\vee, h_4^\vee) = (\bar{h}_1, h_2, h_3, h_4), \quad \xi = \mathrm{Id};$$
- if $e(h_2)$ connects $v(h_3)$ with $v(\bar{h}_4)$, then
$$(h_1^\vee, h_2^\vee, h_3^\vee, h_4^\vee) = (h_1, \bar{h}_2, h_3, h_4), \quad \xi = \mathrm{Id};$$

- otherwise, let

$$(h_1^\vee, h_2^\vee, h_3^\vee, h_4^\vee) = (h_3, h_4, h_1, h_2),$$

$$\xi(1) = 3, \quad \xi(2) = 4, \quad \xi(3) = 1, \quad \xi(4) = 2.$$

Case (ii) , $h_3 = h_2'$:

- if $v(h_1) = v(h_2)$ and $v(\bar{h}_1) = v(\bar{h}_4)$, then

$$(h_1^\vee, h_2^\vee, h_3^\vee) = (h_1, h_4, \bar{h}_2), \quad h_4^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_2) = v(h_4)$ and $v(h_1) = v(h_2)$, then

$$(h_1^\vee, h_2^\vee, h_3^\vee) = (\bar{h}_4, h_2, h_1), \quad h_4^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_2) = v(h_4)$ and $v(\bar{h}_1) = v(\bar{h}_4)$, then

$$(h_1^\vee, h_2^\vee, h_3^\vee) = (\bar{h}_1, \bar{h}_2, h_4), \quad h_4^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_2) = v(h_4)$, $v(h_1) \neq v(h_2)$ and $v(\bar{h}_1) \neq v(\bar{h}_4)$, then

$$(h_1^\vee, h_2^\vee, h_3^\vee) = (h_4, \bar{h}_1, h_2), \quad h_4^\vee = h_2^{\vee\prime},$$

$$\xi(1) = 1, \quad \xi(2) = 1, \quad \xi(3) = 3, \quad \xi(4) = 4;$$

- otherwise, let

$$(h_1^\vee, h_2^\vee, h_3^\vee) = (h_2, h_4, h_1), \quad h_4^\vee = h_2^{\vee\prime},$$

$$\xi(1) = 2, \quad \xi(2) = 4, \quad \xi(3) = 1, \quad \xi(4) = 4.$$

Case (iii) , $h_4 = h_2'$:

- if $v(h_2) = v(h_3)$ and $v(\bar{h}_2) = v(\bar{h}_1)$, then

$$(h_1^\vee, h_2^\vee, h_4^\vee) = (h_3, h_2, \bar{h}_1), \quad h_3^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_3) = v(h_1)$ and $v(\bar{h}_2) = v(\bar{h}_1)$, then

$$(h_1^\vee, h_2^\vee, h_4^\vee) = (h_1, \bar{h}_3, h_2), \quad h_3^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_3) = v(h_1)$ and $v(h_2) = v(h_3)$, then

$$(h_1^\vee, h_2^\vee, h_4^\vee) = (\bar{h}_1, \bar{h}_2, h_3), \quad h_3^\vee = h_2^{\vee\prime}, \quad \xi = \mathrm{Id};$$

- if $v(\bar{h}_3) = v(h_1)$, $v(\bar{h}_2) \neq v(\bar{h}_1)$ and $v(h_2) \neq v(h_3)$, then

$$(h_1^\vee, h_2^\vee, h_4^\vee) = (\bar{h}_2, h_3, h_1), \quad h_3^\vee = h_2^{\vee\prime},$$

$$\xi(1) = 1, \quad \xi(2) = 3, \quad \xi(3) = 3, \quad \xi(4) = 4;$$

- otherwise, let

$$(h_1^\vee, h_2^\vee, h_4^\vee) = (h_3, h_1, h_2), \quad h_3^\vee = h_2^{\vee\prime},$$

$$\xi(1) = 3, \quad \xi(2) = 1, \quad \xi(3) = 1, \quad \xi(4) = 2.$$

Case (iv) , $h_3 = \overline{h_2'}$:

$$(h_1^\vee, h_2^\vee, \overline{h_4^\vee}, \overline{h_3^\vee}, U_1^\vee, \ldots, U_m^\vee) = \mu(h_1, h_2, \bar{h}_4, \bar{h}_3, U_1, \ldots, U_m)$$

where the right hand side is defined by case (iii) .

Case (v) , $h_4 = \overline{h_2'}$:

$$(h_1^\vee, h_2^\vee, \overline{h_4^\vee}, \overline{h_3^\vee}, U_1^\vee, \ldots, U_m^\vee) = \mu(h_1, h_2, \bar{h}_4, \bar{h}_3, U_1, \ldots, U_m)$$

where the right hand side is defined by case (ii) .

In the above, $h_2^{\vee\prime}$ is always taken to be in $(X_1 \cdots X_n)_{h_1^\vee, h_2^\vee}$. It is a straightforward check that μ defines a pairing between elements of F, and moreover,

$$((X_1 \cdots X_n)_{h_1,h_2})_{h_3,h_4} = -((X_1 \cdots X_n)_{h_1^\vee, h_2^\vee})_{h_3^\vee, h_4^\vee}.$$

Hence, the sum in (5) corresponding to $f \in F$ is the negative of the sum in (5) corresponding to $\mu(f)$. This completes the proof of Theorem 3.

Appendix: Strong homotopy Lie bialgebras

In this appendix, we give the definition of strong homotopy Lie (super) bialgebras.

A strong homotopy Lie bialgebra in the category of super vector spaces over a field $\Bbbk$ of characteristic 0 consists of the data of:

- a super vector space $V = V_+ \oplus V_-$ over $\Bbbk$;
- an odd linear map $\alpha_{m,n} : (\Pi V)^{\otimes n} \to (\Pi V)^{\otimes m}$ for each ordered pair of positive integers m, n.

These data are required to satisfy the following conditions:

- $\alpha_{m,n}$ is commutative and cocommutative for all m, n;
- one has:
$$\sum_{T \in T(m,n)} \alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} = 0$$
for all m, n.

We remind that the set $T(m,n)$ of flowcharts was defined in Section 1, and for each $T \in T(m,n)$,

$$\alpha_{o(t(T)),i(t(T))} \circ_T \alpha_{o(s(T)),i(s(T))} : (\Pi V)^{\otimes n} \to (\Pi V)^{\otimes m}$$

is the composition of $\alpha_{o(t(T)),i(t(T))}$ and $\alpha_{o(s(T)),i(s(T))}$ according to T.

The above definition comes from the theory of Koszul duality of dioperads, cf. [G]. The reader may refer to [M] for further details.

Acknowledgments

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